Mathematics and Music: Using Group Theory to Qualify N-Note Tonal Systems

By

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Abstract

In this senior thesis we examine the relationship between math, specifically group theory, and tonal music systems. In particular, we examine the 12-note tonal system of western music and identify some of its fundamental properties. From there we establish criteria that an alternative tonal system must meet to be considered an acceptable alternative to the 12-note tonal system. We build a mathematical model for n-note tonal systems using cyclic groups of order n. Ultimately we conclude that in order for an n-note tonal system to be acceptable, the system must be composed of n notes where n is divisible by 4, but not divisible by 8.

1 Introduction

Within this project we set criteria for acceptable alternatives to the 12-note tonal system and determine which n-note tonal systems meet these criteria. In section 2 we introduce the basic musical terms we will use throughout the paper. In section 3 we review literature that introduces cyclic groups to the study of tonal systems. We use section 4 to introduce the basic concepts of cyclic groups and how they correspond to the 12-note tonal system. Section 5 sets out our criteria for alternative tonal systems. Then in section 6 we apply our techniques to the n-note tonal system. We show that in order to meet our criteria the number of notes in the n-note tonal system must be divisible by 4 and not divisible by 8. In addition, we show such a system necessarily results in a diatonic major scale consisting of $\frac{n}{2} + 1$ notes. In section 7 and 8 we examine the 20-note and 16-note tonal systems respectfully. Finally, we explain our conclusions in section 9.

2 Musical Terms

In this paper we examine several musical concepts including: the semitone, the diatonic major and minor scale, the octave, the perfect fifth, the perfect fourth, the major third, the minor third and the major and minor triads. We will use these terms throughout the paper to explain our criteria and important features of tonal systems.

- "The semitone is the smallest interval in use in the Western music tradition. [...] The semitone is represented on the piano keyboard by the distance between any two immediately adjacent keys, whether white or black" [11]. Since the semitone is the smallest interval, every interval can be described as being comprised of a certain number of semitones. A half-step (H) is equivalent to one semitone and a whole-step (W) is equivalent to two semitones.

- "The diatonic major scale is a scale with seven different pitches that are adjacent to one another on the circle of fifths" [5]. The diatonic major scale follows the pattern WWHWWWH. For example, starting at C, a whole step above is D. Then a whole step above D is E; then a half step above E is F. Continuing to follow the pattern, we find that the white keys on the keyboard from middle to high C form the C-major diatonic scale. The diatonic minor scale follows the pattern WHWWHWW. The white keys on the keyboard starting at A are an example of a diatonic minor scale. Major and minor diatonic scales were the basis of music written from 1600-1900.

- The octave (8ve) is "an interval bounded by two pitches with the same pitch names and the higher of whose frequencies is twice the lower" [10]. The distance of an 8ve can be seen on a keyboard when moving from one C to the next C and is composed of 12 semitones. Note that the outer limits of the diatonic major and minor scales are an 8ve apart.

- The perfect fifth (P5) is the interval of 7 semitones [9]. The interval P5 sounds consonant and appears in the major and minor triads. The distance from C to G ascending on the keyboard is an example of a P5.

- The perfect fourth (P4) is the interval of 5 semitones [9]. The distance from G to C ascending is an example of the interval P4. Notice that P4 complements P5 which means P5 + P4 = 8ve and equivalently 7 semitones + 5 semitones = 12 semitones.

- The major third (M3) is the interval of 4 semitones [9]. The distance from C to E ascending is an example of the interval M3. Note that M3 appears as the bottom interval of the major triad.

- The minor third (m3) is the interval of 3 semitones [9]. The distance from E to G ascending is an example of a m3. Notice that m3 compliments M3 which means M3 + m3 = P5 and equivalently 4 semitones + 3 semitones = 7 semitones. Note that m3 appears as the bottom interval of the minor triad.

- A chord is "three or more pitches sounded simultaneously" [2]. The most basic chords in tonal music are the major and minor triads. A triad is "a chord consisting of three pitches, the adjacent pitches being separated by a third" [12]. A major triad consists of a root note and the notes that are a M3 and a P5 above the root note. The notes C, E, and G sounded together are an example of a major triad because the interval from C to E is a M3 and the interval from C to G is a P5. A minor triad consists of a root note and the notes that are a m3 and a P5 above the root note. The notes C, Eb, and G sounded together are an example of a minor triad because the interval from C to Eb is a m3 and the interval from C to G is a P5.

- Circle of Fifths: The circle of fifths is "the arrangement in a closed circle of all 12 pitch names (notes) in such a way that, when proceeding clockwise along the circle, any pair of adjacent pitch names represents the interval of a perfect fifth [see figure 1]. Thus, C to G is a perfect fifth, as are G to D, D to A, A to E, E to B, etc" [3] and thus P5 generates all 12 tones without repetition. In the circle of fifths, each of the notes corresponds to the diatonic major scale starting at that note. Connected regions of seven notes in the circle of fifths comprise the notes in a diatonic scale. The major scales take their names from the second note in the region when notes are counted in the clockwise direction. This note is the tonic note, the one that the scale starts on when played. Scales that are next to each other on the circle of fifths have similar structure; they only differ from one another by one tone.

3 Literature Review

In preparation for this research we examined literature concerning n-note tonal groups and their properties. Most of these papers were written from a musical perspective, and we hope to expand on their ideas.

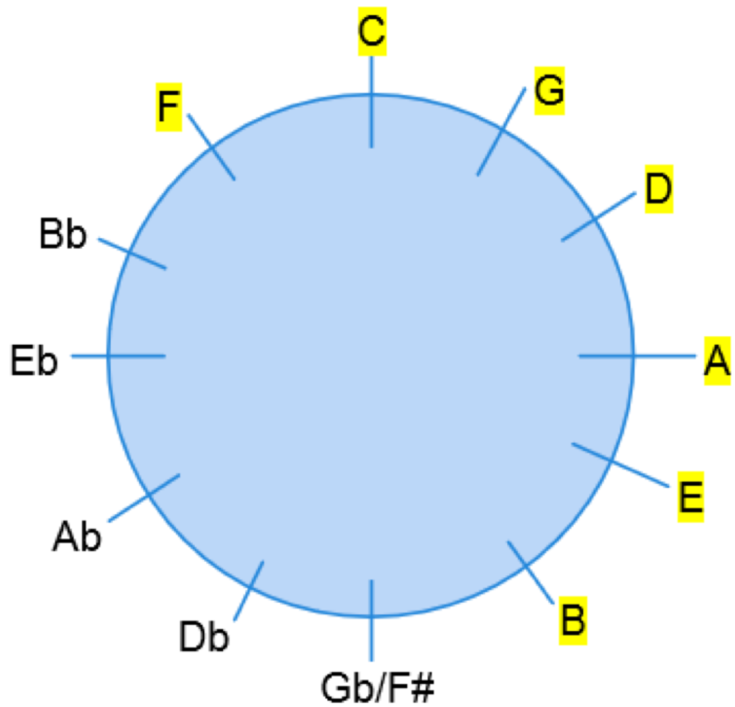

Figure 1: Circle of Fifths (Diatonic Notes of C Major Highlighted)

First, we examined the article “The Group-Theoretic Description of 12-Fold and Microtonal Pitch Systems.” by Gerald J. Balzano [1]. Balzano was the first to introduce cyclic groups to the study of tonal systems. He identified that the generalized P5 is a generator of cyclic groups; therefore an n-note microtonal system must have a generator, p, that can serve as its P5. He believed that the generator p must be able to be partitioned into two relatively prime integers whose product is equal to n. Then he chose to let the bigger of the two relatively prime integers serve as his M3, and the smaller of the two integers serve as his m3. For example, in his 20-note tonal system he chose the generalized P5 = 9, and from there found the generalized M3 = 5 and generalized m3 = 4. Balzano also introduced the concept of the $F \rightarrow F\sharp$ property which is defined as “adjacent scales differ in a single note changed by one semitone” [1]. The $F \rightarrow F\sharp$ property is desirable because it means if two tones are next to each other on the circle of fifths then their keys are similar. They only differ by one note and even then that difference is as small as possible. Balzano also introduced the Balzano diagram, which is a Cayley diagram with generators P5, M3, and m3. Balzano explained that a reasonable system should have a diatonic major scale consistent with the $F \rightarrow F\sharp$ property and then he should be able to trace the notes in the scale as a connected region in the Balzano diagram without intersection.

Paul F. Zweifel added to Balzano's work in his paper "Generalized Diatonic and Pentatonic Scales: A Group-Theoretic Approach" [13] and concluded that Balzano's 9-note diatonic scale for the 20-note tonal system actually functioned more as a pentatonic scale than a diatonic scale and he proposed an 11-note diatonic major scale. Zweifel argued that Balzano's claim that the generator of the group must be able to be partitioned into two relatively prime integers whose product is equal to n was too strict. He felt that the integers merely needed to be relatively prime and add to p, but that they did not need to have a product of n.

Next, we considered the article "Balzano and Zweifel: Another Look at Generalized Diatonic Scale" by Mark Gould [7] , in which Gould gave an alternative opinion to the 11-note diatonic scale that Zweifel proposed for the 20-note tonal system. Gould believed that there are more tonal systems that would be similar to the 12-note tonal system than just the 20-note tonal system, and he set out a list of criteria that possible alternative tonal systems must meet. However, his system involved individually checking every tonal system and every possible diatonic scale for each tonal system. We were able to describe a more refined system that does not involve the process of elimination.

Balzano, Zweifel and Gould all are music theorists who introduced group theory into their work. Therefore, their papers are all written primarily from a musical audience. We also considered an article written about music concepts, but from a mathematical point of view: the article "Musical Actions of Dihedral Groups" by Alissa S. Crans, Thomas M. Fiore, and Ramon Satyendra [4]. Crans, Fiore, and Satyendra discussed the 12-note tonal system and the different transposition and inversion techniques and how they mirrored musical operations. They considered the mapping of the major chord to its parallel minor, which is defined as the minor chord that starts on the same note as the major chord (for example C,E,G →C,E♭,G). They also investigated the mapping of the major chord to its relative minor, which is defined as the minor key that has the same amount of sharps and flats as the major key (for example C,E,G → A,C,E). Although our paper is not specifically tied to the work of Crans, Fiore, and Satyendra, their work does pose an interesting question we could examine in further studies; in particular, do the n-note tonal systems have similar operation maps to the 12-note tonal system?

4 The Mathematical Model

This section contains a brief introduction to the group $\mathbb{Z}_n$ which we will use to model n-note-note tonal systems. The group consists of the integers $\{0, 1, 2, \ldots, n-1\}$ and uses the operation of addition modulo n (see definition 1 to combine group elements.

Definition 1. *Consider the integers a and b and positive integer n. We define $a \mod n$ to be the remainder upon dividing a by n. That is, if we use the division algorithm to write a as $qn + r$ with $0 \leq r < n$ then, $a \mod n = r$.*

For example, $4 \mod 2 = 0$ because $4 = 2 \cdot 2 + 0$. Another example is $13 \mod 4 = 1$ because $13 = 3 \cdot 4 + 1$. Modular arithmetic is concerned with the remainder when dividing a number, not the amount of times the number can be divided.

To form our model we relate sets of numbers to their subsequent pitch classes in the music world. For example, we assign the numbers in the group $\mathbb{Z}_{12}$ (the integers from 0 to 11) to each pitch on the 12-tone chromatic scale, meaning that 0 corresponds to the note C, 1 to C♯, 2 to D, etc. (as seen in figure 2).

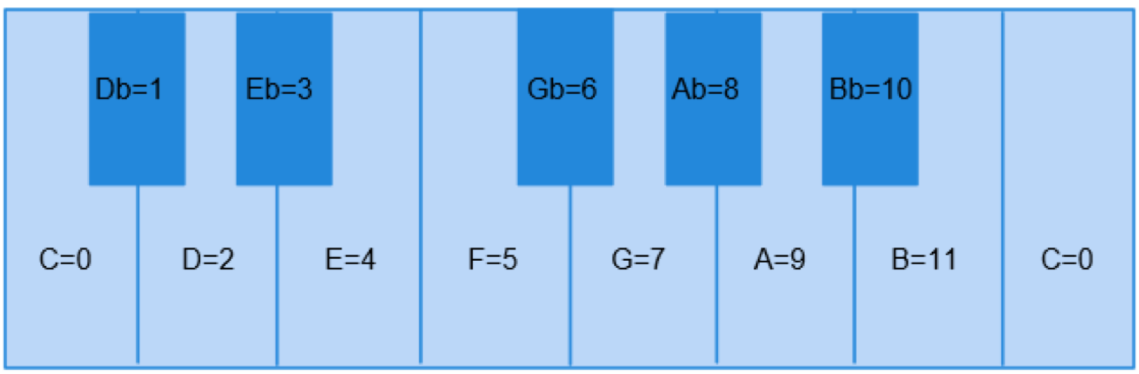

Figure 2: 12-Note Keyboard

Subsequently we also examine sets of two integers in $\mathbb{Z}_{12}$ and their corresponding intervals, or distances. For example, P5 is the musical equivalent to adding $7 \mod 12$ to a number. Consider the starting pitch G which corresponds to the number 7. To find the P5 above G we take 7 and add $7 \mod 12$: $7 + 7 \mod 12 = 2$, which corresponds to the pitch D. Therefore D is the P5 above G.

Similarly we observe the same correspondence when we look at the other important intervals P4, M3, and m3. A P4 can be found using the same method we showed for P5, except instead of adding $7 \mod 12$, we add $5 \mod 12$. This is because a perfect 4th has to be able to be added to the perfect 5th in order to create an octave. We know a perfect 4th is 5 semitones from the starting pitch because $5+7 \mod 12 = 0$. Consider the note A. Referring to figure 2, we know A corresponds to 9. To find the P4 above A, we find $9+5 \mod 12 = 2$; since 2 corresponds to D we can conclude that D is P4 above A.

We also know that the M3 must be one semitone larger than the m3. Therefore, the M3 can be found by adding $4 \mod 12$ to a starting pitch, and the m3 can be found by adding $3 \mod 12$. For example, we can find the M3 above B♭ = 10 by calculating $10+4 \mod 12 = 2$; therefore D=2 is a M3 above B♭ = 10. Similarly $10+3 \mod 12 = 1$; therefore D♭ is a m3 above B♭. We can also see that M3 + m3 = P5 because $4+3 \mod 12 = 7$.

The groups $\mathbb{Z}_n$ fall into a category of groups known as cyclic groups, which are defined in terms of generators. Definition 2 gives the full explanation of a cyclic group and theorem 1 explains how to identify the generators for the groups $\mathbb{Z}_n$. The generators in theorem 1 are defined in terms of greatest common divisors, and theorem 2 gives some important properties of greatest common divisors.

Definition 2. *An additive group G is called cyclic if there exists an $a \in G$ such that $G = \{ka | k \in \mathbb{Z}\}$ where $ka = \underbrace{a + a + \cdots + a}_{k \text{ times}}$. In this case we call a a generator of G.*

Theorem 1 (Corollary 4.3 [6])**.** *An integer is a generator of $\mathbb{Z}_n$ if and only if $gcd(n, k) = 1$.*

Theorem 2 (Theorem 0.2 [6])**.** *For any nonzero integers a and b, there exist integers s and t such that $gcd(a, b) = as + bt$. Moreover, $gcd(a, b)$ is the smallest positive integer of the form $as + bt$.*

As a consequence of theorem 2 we can show the integers a and b are relatively prime, i.e. $\gcd(a, b) = 1$, by simply finding integers s and t such that $as + bt = 1$.

Finally, in our analysis of n-note tonal systems we evaluate Cayley graphs of $\mathbb{Z}_n$, which are used to visually represent these groups and their generators. The Balzano diagram introduced in section 3 is an example of Cayley graphs for the group $\mathbb{Z}_n$ using the generating set $\{\mathrm{P5}, \mathrm{M3}, \mathrm{m3}\}$. Note that Cayley graphs can be applied to any group, but we are interested specifically in Cayley graphs for the group $\mathbb{Z}_n$. We adapted definitions 3 and 4 from the more general definition found on pages 497-498 of Gallian [6]. The musical examples following the definitions should clarify their meanings.

Definition 3. *We say that a set S is a generating set for $\mathbb{Z}_n$ whenever every element in $\mathbb{Z}_n$ can be formed from repeated combinations of elements in S. In particular, any set S containing a generator for $\mathbb{Z}_n$ will be a generating set for $\mathbb{Z}_n$. However, this condition is not necessary as any set containing elements that sum to a generator would also generate.*

Definition 4. *Let S be a generating set for $\mathbb{Z}_n$. The elements of $\mathbb{Z}_n$ form the vertices of the Cayley graph with generating set S. For any x and y in $\mathbb{Z}_n$, we draw an edge from x to y if and only if $x + s \mod n = y$ for some s in S.*

Example 1. *In $\mathbb{Z}_{12}$ if we take $S = \{P5\} = \{7\}$, we get the circle of fifths as our Cayley graph, as seen in figure 3.*

Example 2. *If we take $S = \{P5, M3, m3\} = \{7, 4, 3\}$ as our generating set for $\mathbb{Z}_{12}$, we get the Balzano Diagram associated with $\mathbb{Z}_{12}$. In order to create this Cayley Graph, we assign a direction to each generator, as seen in figure 4.*

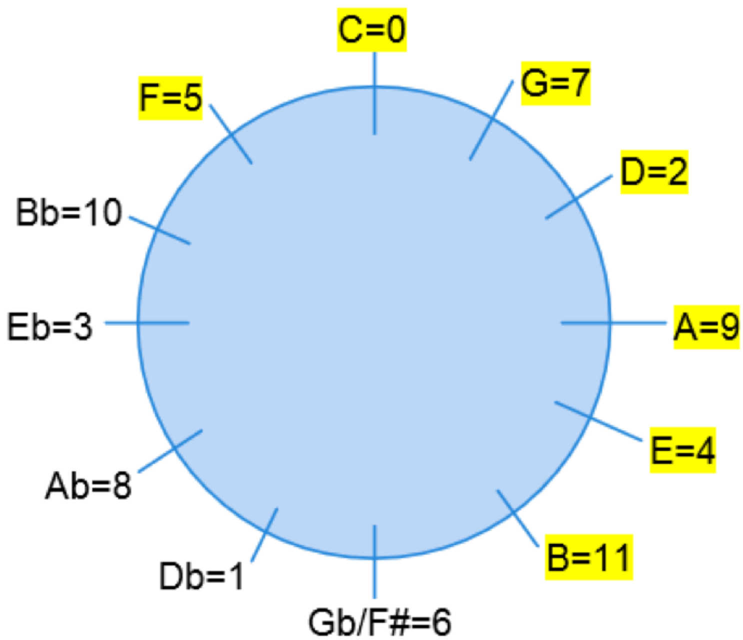

Figure 3: Simple Cayley Graph with P5 as the Generator

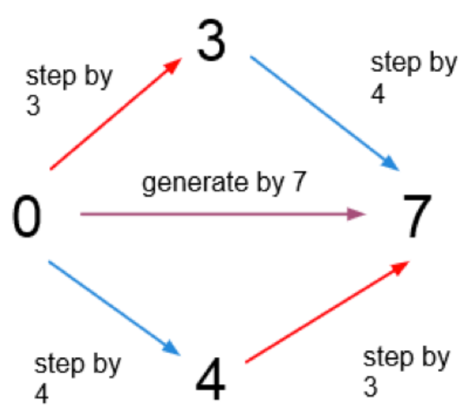

Figure 4: Creating a Cayley Graph for a 12-Note Tonal System

First consider the action of the P5 $= 7$ *arrows. When we start at zero, the next number horizontally is 7, then the next P5 arrow would bring us to 2 because* $7 + 7 \mod 12 = 2$. *Next we would add* $2 + 7 \mod 12 = 9$, *then* $9 + 7 \mod 12 = 4$, *etc. Therefore, following the horizontal line is the same as traveling around the circle of fifths. The P5* $= 7$

generator can then be broken down into steps of M3 = 4 and m3 = 3. We use these numbers as additional elements of our generating set, so our Cayley graph will be a grid instead of a circle. After incorporating the M3 and m3 arrows into our diagram we have the complete Cayley graph with generating set $S = \{P5,\ M3, m3\} = \{7, 4, 3\}$ *as seen in figure 5. It should be noted that any repeated numbers in this Cayley graph are actually the same number because the Cayley graph is a flattened version of a twisted torus.*

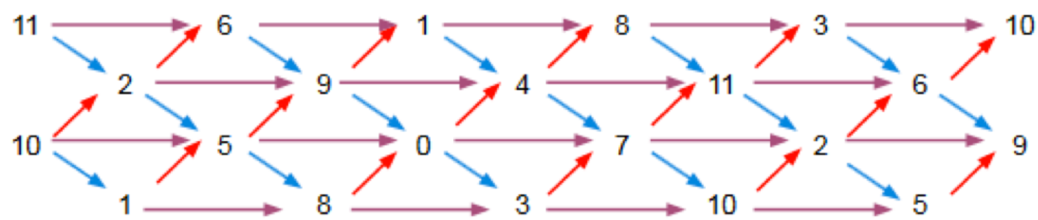

Figure 5: Cayley Graph for 12-Note Tonal System

The Cayley graph in Example 8 can be used to see visually how a chordal system is related to a diatonic major scale. Mathematically this means taking the generator of 7 and dividing it into 3 and 4. Musically this represents taking the P5 and showing how it is composed of a M3 and a m3.

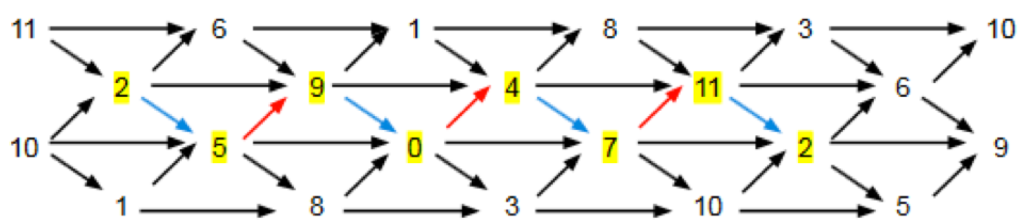

Figure 6: Highlighting the Diatonic Major Scale for the 12-Note Tonal System

Looking at the Cayley graph in figure 6, if we start at 2, and follow the path starting with a m3 and alternating between m3 and M3 we can trace a diagonal path across the

Cayley diagram that generates all the tones in the diatonic major scale without repetition. This creates the traditional 7 note diatonic major scale. The ability to trace the diatonic scale as a connected region in the Cayley graph with generating set $\{\mathrm{P5}, \mathrm{M3}, \mathrm{m3}\}$ has been identified by Balzano, Zwiefel, and Gould as an important property of any n-note tonal system that mirrors western 12-tone music.

5 Criteria of an Alternative Tonal Systems

After examining the 12-note tonal system and determining its important features, we produced the following criteria that qualify an n-note tonal system as an acceptable alternative to the western 12-note system.

1. The tonal system must be of size n where n is even.

2. The tonal system must have equivalents of P5, P4, M3, and m3. We choose the intervals to be as close to the 12-note tonal system equivalents as possible. Specifically:

   (a) $\mathrm{P5} = \frac{n}{2+1}$. In theorem 3 we show that if $\mathrm{P5} = \frac{n}{2} + 1$ generates a circle of fifths, then n must be a multiple of 4.

   (b) $\mathrm{P4} + \mathrm{P5} = 8\mathrm{ve}$.

   (c) $\mathrm{M3} = \frac{n}{4} + 1$.

   (d) $\mathrm{M3} + \mathrm{m3} = \mathrm{P5}$.

3. The tonal system must have a diatonic major scale with the $F \to F\sharp$ property. This implies that the diatonic major scale must contain $\frac{n}{2}+1$ notes. We will prove this later in this paper in theorem 4.

4. All notes within the diatonic major scale must be able to be generated with altering steps of $\frac{n}{4}$ and $\frac{n}{4}+1$. In theorem 5 we prove this is possible if and only if n is divisible by 4 but not by 8.

6 Generalizing for the n-note Tonal System

In this section we consider which n-note tonal systems meet the criteria laid out in section 5. We assume that n is an even integer. First we determine when our generalized P5, $\frac{n}{2}+1$, generates a generalized circle of fifths.

Theorem 3. *We have a generalized circle of fifths if and only if n is divisible by 4.*

Proof. We will have a generalized circle of fifths if and only if P5 $= \frac{n}{2}+1$ generates $\mathbb{Z}_n$. First suppose n is divisible by 4, that is $n = 4s$ for some integer s. Then $\frac{n}{2}+1 = 2s+1$ generates $\mathbb{Z}_n$ since

$$\begin{aligned} sn + (1-2s)(2s+1) \mod 4s &= s(4s) + (1-2s)(2s+1) \mod 4s \\ &= s(4s) + 1 - (4s)s \mod 4s \\ &= 1. \end{aligned}$$

Therefore, by theorem 2, n and $2s+1$ are relatively prime. Thus theorem 1 guarantees that $2s+1 = \frac{n}{2}+1$ generates $\mathbb{Z}_n$, yielding a generalized circle of fifths.

If $n \neq 4s$ for some s then $n = 2k$ where k is odd, but then $\frac{n}{2} + 1 = k + 1$ is even, so $\gcd(n, \frac{n}{2} + 1) \geq 2$ and $\frac{n}{2} + 1$ does not generate $\mathbb{Z}_n$, so that $2s + 1 = \frac{n}{2} + 1$ cannot generate a generalized circle of fifths. □

Next we check that the n-note tonal system with n a multiple of 4 will allow a diatonic major scale (i.e. a connected region on the generalized circle of fifths) with the $F \to F\sharp$ property. Theorem 4 shows this is possible and leads to a diatonic major scale containing $\frac{n}{2} + 1$ notes.

Theorem 4. *The n-note tonal system with n divisible by 4 and generalized P5 $= \frac{n}{2} + 1$ has a diatonic scale (i.e. a connected region in the generalized circle of fifths with the $F \to F\sharp$ property) if and only if the connected region is of length $\frac{n}{2} + 1$ notes.*

Proof. Let k be the number of notes in the diatonic major scale. Assume we start the diatonic scale at C=0. Then the last note is $(k - 1)$ steps past zero on the circle of fifths, so it is $(k-1)\left(\frac{n}{2} + 1\right)$. The next note should be $(k)\left(\frac{n}{2} + 1\right)$. If the system has the $F \to F\sharp$ property, then $k\left(\frac{n}{2} + 1\right) - 0 \mod n = 1$. Thus, $k = \left(\frac{n}{2} + 1\right)^{-1} \mod n$. Observe:

$$\begin{aligned}
\left(\frac{n}{2} + 1\right)\left(\frac{n}{2} + 1\right) &= \frac{n^2}{4} + n + 1 \mod n \\
&= n\left(\frac{n}{4}\right) + n + 1 \mod n \\
&= 1,
\end{aligned}$$

so that $\left(\frac{n}{2} + 1\right)^{-1} \mod n = \left(\frac{n}{2} + 1\right)$ and$k = \frac{n}{2} + 1$.

Now we consider a connected region containing $\frac{n}{2}+1$ notes. If we begin our connected region at C=0, the last note would be $\left(\frac{n}{2}\right)\left(\frac{n}{2}+1\right) \mod n$. The note we would gain with the shift would be $\left(\frac{n}{2}+1\right)\left(\frac{n}{2}+1\right) \mod n$. Since

$$\left(\frac{n}{2}+1\right)\left(\frac{n}{2}+1\right) \mod n = \left(\frac{n}{2}+1\right)\left(\frac{n}{2}+1\right)^{-1} \mod n = 1,$$

we have the $F \to F\sharp$ property. $\square$

Now that we know a group must be of order 4 in order to be generated by $\frac{n}{2}+1$, we prove that the system's order must not be divisible by 8. We have only the final criteria of section 5 left to consider, specifically that a diatonic scale can be generated by alternating steps of a generalized M3 and m3. Note that we choose to use the C-major scale, but it can be shown that this process works for any diatonic major scale. We show in this next theorem that this is only possible when n is not divisible by 8.

Theorem 5. *Let n be a positive integer that is divisible by 4, and let the C-major scale be the unique collection of $\frac{n}{2}+1$ consecutive notes on the generalized circle of fifths , i.e. the Cayley graph generated by $\frac{n}{2}+1$, beginning at the note $-\left(\frac{n}{2}+1\right) \mod n$. This C-major scale can be generated by altering steps of $\frac{n}{4}$ and $\frac{n}{4}+1$ if and only if n is not divisible by 8.*

Proof. First, let $s = \frac{n}{4}$. Then P5 $= 2s+1$, M3 $= s+1$, m3 $= s$, and $n = 4s$. On the circle of fifths, the C-major scale starts at $-\left(\frac{n}{2}+1\right) \mod n$. This simplifies as follows:

$$\begin{aligned} -\left(\frac{n}{2}+1\right) \mod n &= n - \left(\frac{n}{2}+1\right) \mod n \\ &= \frac{n}{2} - 1 \mod n \\ &= 2s - 1 \mod n. \end{aligned}$$

Note that $2s-1 =$ P4 because P5 + P4 = 8ve, and $(2s+1)+(2s-1) \mod 4s = 0$. The C-major diatonic scale appears on the circle of fifths in this order:

- 1st note: $2s-1$;
- 2nd note: $(2s-1)+(2s+1) \mod 4s = 0$;
- 3rd note: $2s+1$;
- 4th note: $(2s+1)+(2s+1) \mod 4s = 4s+2 \mod 4s = 2$;

$$\vdots$$

- kth note: $\begin{cases} 2s+(k-2) \mod 4s & \text{if } k \text{ is odd} \\ k-2 & \text{if } k \text{ is even} \end{cases}$

$$\vdots$$

- (2s+1)st note: $2s+(2s+1-2) \mod 4s = 4s-1 \mod 4s = -1 \mod n$ (because k is odd).

We need to start our connected region on the three-generator Cayley graph in the middle of the region as it appears in the generalized circle of fifths. Only by beginning there can we return to the note at which we started. If we do not start in the middle then we would generate a note not in the diatonic major scale before we generated all the notes in the scale. Consider the 12-note tonal group. Looking at figure 7, we see that if we start in the middle of the scale, in this case at D = 2, the red arrow indicates the first m3 step to F = 5. We then can follow the same path we would trace in the Cayley graph on the circle of fifths.

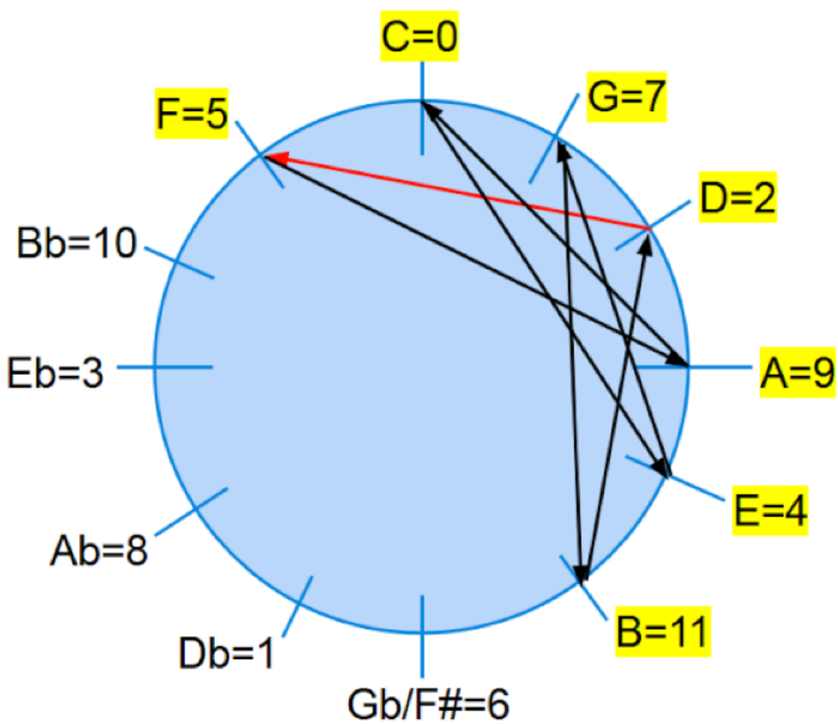

Figure 7: Tracing out the Diatonic Major Scale on the Circle of Fifths

The middle of the connected region of $2s+1 = \frac{n}{2}+1$ notes occurs at the $\frac{2s+1+1}{2} = (s+1)$st note in the region. Therefore, we should start our path on the

Cayley graph with generators P5,M3, and m3 at the note

$$\begin{cases} 2s + ((s+1) - 2) \mod 4s & \text{if } s \text{ is even} \\ s + 1 - 2 \mod 4s & \text{if } s \text{ is odd} \end{cases}$$

$$= \begin{cases} 3s - 1 & \text{if } s \text{ is even} \\ s - 1 & \text{if } s \text{ is odd.} \end{cases}$$

Recall that we are trying to show if s is odd, we can generate all of the pitches of the diatonic major scale without repetition as a path on the Cayley graph with the generating set $\{\text{P5}, \text{M3}, \text{m3}\}$; however we cannot do so if s is even.

Case 1: Suppose that s is odd. Therefore, the connected region on the Cayley graph starts at the note $s-1$. We need to either start by following an m3(s) edge or a M3($s+1$) edge. If we begin by following a m3 edge, then we move to the note $(s-1)+s \mod 4s = 2s-1$, but if we follow a M3 edge, then we move to the note $(s-1)+(s+1) \mod 4s = 2s$. We must follow the m3(s) edge because $2s-1$ is in the diatonic major scale and $2s$ is not in the diatonic major scale. If $2s$ were in the diatonic major scale then it would be the $(2s+2)$nd note, but the diatonic scale stops at the $(2s+1)$st note. Thus in this case, we must start by following a m3(s) edge, starting at the note $s-1$. Note that all the odd edges in our sequence will be m3, while all even edges will be M3.

Consider the notes in our connected region on the Cayley graph with generators $\{\text{P5}, \text{M3}, \text{m3}\}$:

- First note: $s - 1$.
- Second note: $(s-1) + s \mod 4s = 2s - 1$.
- Third note: $(2s-1) + (s+1) \mod 4s = 3s$. We know $3s$ is in the diatonic major scale because it is P5 above $s-1$, which is the $(s+1)$st note in the diatonic major scale to appear on the generalized circle of fifths, so $3s$ is the $(s+2)$nd note, which has to be before the $(2s+1)$st note unless $s = 1$, then it is the starting note and we stop here.
- Fourth note: $3s + s = 4s \mod 4s = 0$. We know 0 is a note in the diatonic major scale.
- Fifth note: $0 + (s+1) \mod 4s = s + 1$. We know $s+1$ is in the diatonic because it is P5 above 3s, so it is the $(s+3)$rd note in the diatonic major scale as it appears on the generalized circle of fifths, we know this is before (2s + 1)st note unless s = 2, but s cannot equal 2 because s is odd.

$$\vdots$$

- Note $2s$: $(2s-2) + s \mod 4s = 3s - 2$.
- Note $2s+1$: $(3s-2) + (s+1) \mod 4s = 4s - 1 \mod 4s = -1$, which is the $(2s+1)$st note in the diatonic scale as it appears on the generalized circle of fifths.
- Note $2s+2$ (to get back to $s-1$): $-1 + s \mod 4s = s - 1$. We have returned to our starting point, collecting all of the notes in the diatonic major scale along the way.

Therefore, we can generate all of the notes in the diatonic major scale using combinations of m3 and M3 (or s and $s+1$) if s is odd. When s is odd n is not divisible by 8 because $n = 4s$.

Case 2: Suppose s is even and the connected region on the Cayley graph starts at $3s-1$. Once again, we either need to follow a m3(s) edge or follow a M3($s+1$) edge. So if we start by following a m3 edge then we start with the note $(3s-1)+s \mod 4s = 4s-1 \mod 4s = -1$. Otherwise, if we start by following a M3 edge, then we start with the note $(3s-1)+(s+1) \mod 4s = 4s \mod 4s = 0$. The notes 0 and -1 are the second and last notes in the diatonic scale respectfully, thus we need to consider both cases. As in case 1, we will consider the diatonic scale in the order it appears on the generalized circle of fifths.

Case a: Suppose we begin by following an m3 edge, obtaining the following sequence of notes.

- First note: $3s-1$;
- Second note: $(3s-1)+s \mod 4s = 4s-1 \mod 4s = -1$;
- Third note: $-1+(s+1) \mod 4s = s$;
- Fourth note: $s+s \mod 4s = 2s$ which is outside the diatonic scale because it is the $(2s+2)$nd step.

Therefore, beginning by following the m3 edge does not allow for the diatonic major scale to appear as a connected region without repetition; it quickly starts generating notes that are not in the C-major scale. For $n = 16$, we can see this illustrated in figure 8.

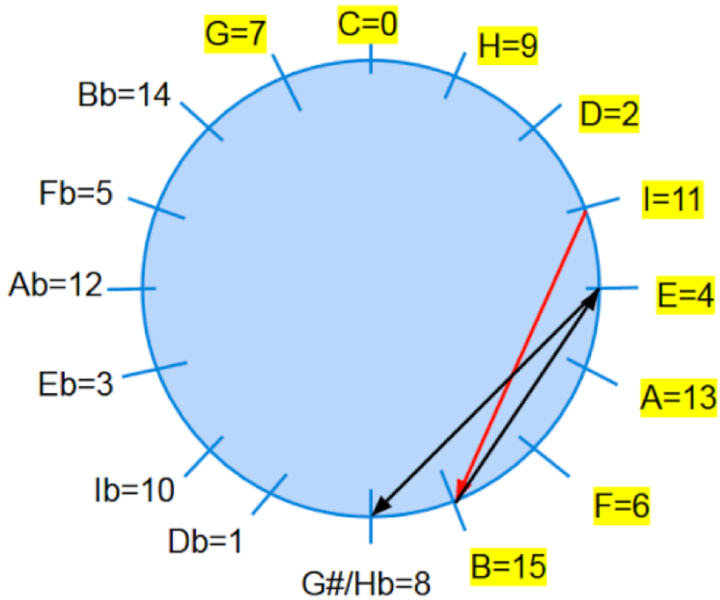

Figure 8: Case a for $n = 16$

Case b: Suppose we begin by following the M3$(s+1)$ edge on the Cayley graph. In this case we create the following sequence of notes. Again we consider the diatonic scale in the order it appears on the circle of fifths.

- First note: $3s - 1$;
- Second note: $(3s-1)+(s+1) = 4s \mod 4s = 0$ which is the 2nd note in the diatonic;
- Third note: $0 + s = s$ which is the $(s+2)$nd note in the diatonic;
- Fourth note: $s + (s+1) \mod 4s = 2s+1$ which is the 3rd note in the diatonic;

- Fifth note: $(2s+1)+s \mod 4s = 3s+1$ which is the $(s+3)$rd note in the diatonic because it is a P5 above s.

$$\vdots$$

- Note $2s-1$:

$$\begin{aligned}(3s-1)+(s-1)(2s+1) \mod 4s &= 3s-1+2s^2-s-1 \mod 4s \\ &= 2s-2+2s^2 \mod 4s \\ &= 2s-2+4s(s^2) \mod 4s \\ &= 2s-2\end{aligned}$$

- Note $2s$: $(2s-2)+(s+1) \mod 4s = 3s-1$ which is our starting note.

Therefore we reached our first note in our sequence before we generated all 2s + 1 notes in the diatonic scale. For $n = 16$, we can see this illustrated in figure 9.

Since both cases a and b failed, we cannot generate all of the notes in the C-major scale when s is even. Therefore we satisfy the criteria set out in section 5, if and only if we choose an n-note tonal system with $n = 4s$ for some odd integer s. Ultimately, n must be divisible by 4 and not divisible by 8 as such a choice would force s to be even.

$\square$

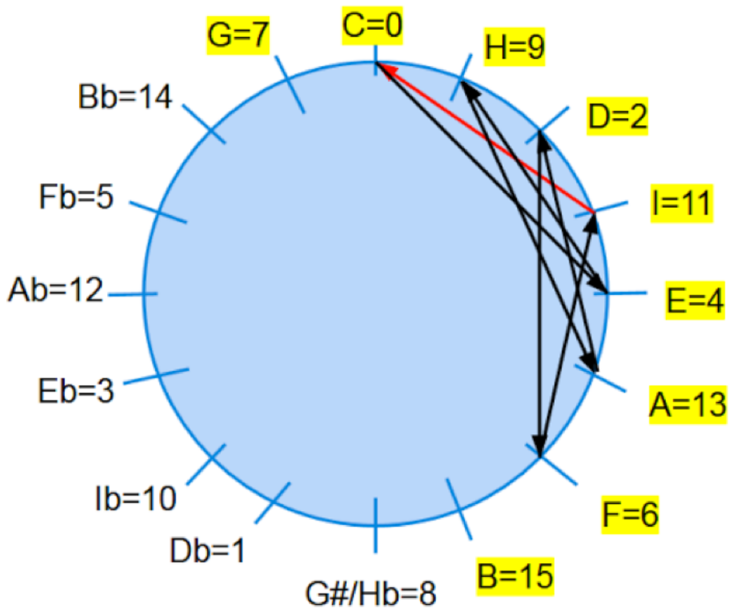

Figure 9: Case b for $n = 16$

7 Extended Example: The 20-Note Tonal System

We model the 20-note tonal system with $\mathbb{Z}_{20}$. Here 0 represents C, 1 represents C$\sharp$, and so on. This alternative tonal system consists of the traditional 12 tones (A-G) as well as new notes as seen in figure 9. This tonal system has a generalized P5 $= \frac{20}{2} + 1 = 11$, a generalized M3 $= \frac{20}{4} + 1 = 6$, and a generalized m3 $= \frac{20}{4} = 5$.

We show the generalized circle of fifths in figure 10. The highlighted notes start at H, which is a P5 below C, and end when we achieve the $F \to F\sharp$ property to form the C-major scale. Notice that if we look at the highlighted region, and drop the first note (H) and pick up the next note on the generalized circle of fifths (H$\sharp$), these notes differ by only one semitone. These highlighted notes make up the diatonic major scale, therefore they are the white keys on our keyboard in figure 11.

As seen in figure 11, the keyboard for the 20-note tonal system looks similar to the one constructed for the 12-note tonal system, with a few extra notes. Notice also that there is still an even number of black keys before the two adjacent white keys (G and H) and then an odd number of black keys after that point.

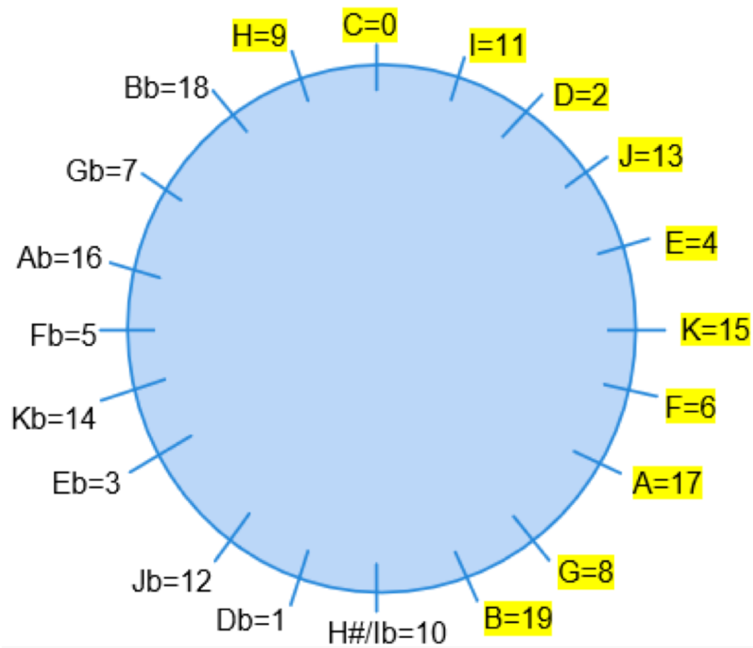

Figure 10: Generalized Circle of Fifths for the 20-Note Tonal System

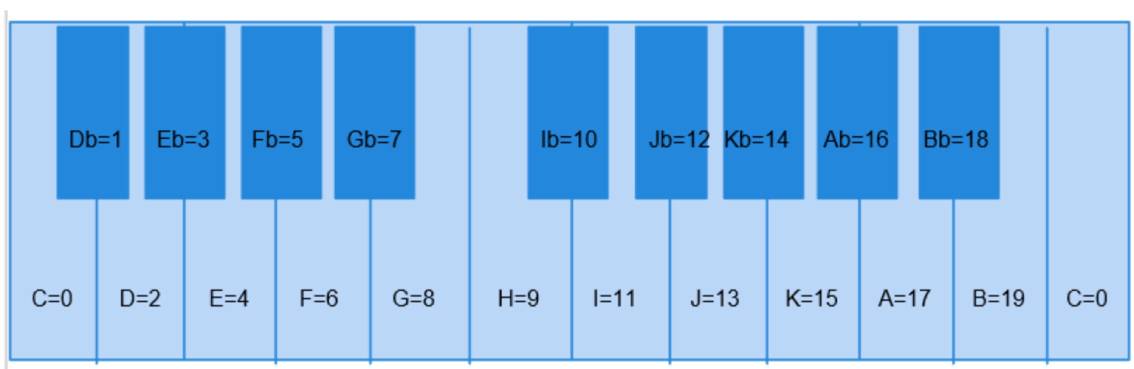

Figure 11: Keyboard for 20-Note Tonal System

Using methods similar to those used to create the Cayley Graph of the 12-note tonal system, we can create the Cayley Graph of the 20-note tonal system with the generating set $\{\mathrm{P5}, \mathrm{M3}, \mathrm{m3}\}$ as seen in figure 12. Once again we are able to trace out the 11-note diatonic major scale without repetition by using alternating steps of m3(5) and M3(6).

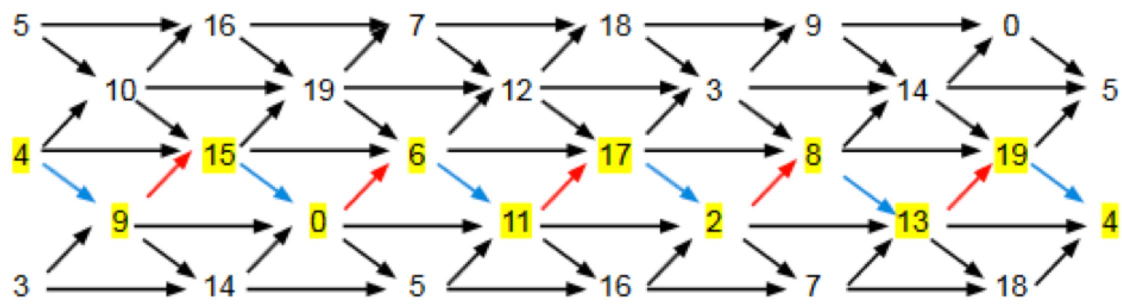

Figure 12: Cayley graph for $\mathbb{Z}_{20}$ with $S = \{11, 6, 5\}$

8 Extended Counter-Example: The 16-Note Tonal System

Next we examine the 16-note tonal system. This system leads to the generalized circle of fifths and keyboard pictured in figures 13 and 14 respectively.

Using the generalized circle of fifths we created in figure 13, we find that the diatonic major scale for the 16-note tonal system are the highlighted keys because they hold to the $F \to F\sharp$ property. Therefore we make these our white keys in figure 14. Notice that M3 for the 16-note tonal system is $\frac{16}{4} + 1 = 5$. This means that M3 above C is Fb, which is not in the diatonic major scale starting at C. This causes the problem that major triads will sound minor and minor triads will sound major. We suspect the reason this happens is because this tonal group did not meet our criteria of an alternative system. Note that the key difference between this keyboard and the keyboards for the 12-tone and 20-note tonal systems is this keyboard has an odd number of black keys together in a row, then an even number. This always happens when the system consists of n tones where n is divisible by 8.

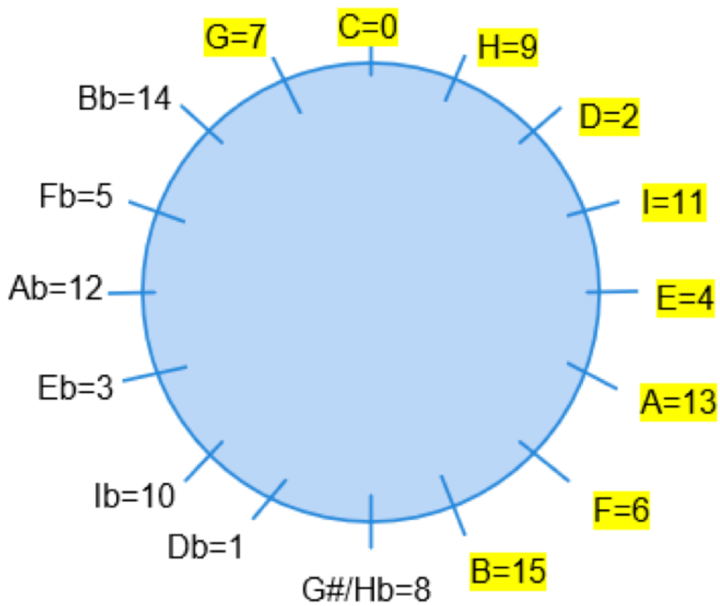

Figure 13: Generalized Circle of Fifths for the 16-Note Tonal System

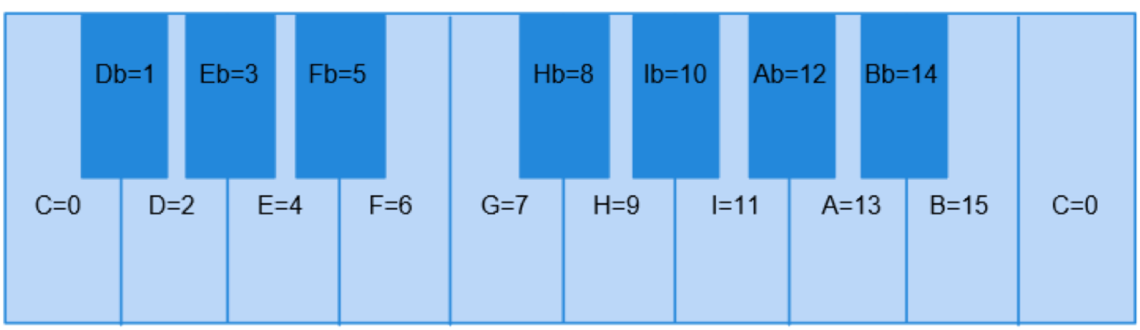

Figure 14: 16-Note Keyboard

We find the Cayley graph with generating set $\{P5, M3, m3\}$ for the 16-note tonal system using the same method as before, except now we have a generalized P5 of 9, which is broken down into the steps 5 and 4 for M3 and m3. Note that this time the larger of the steps is odd, whereas the smaller of the steps is even. This is part of the reason this system fails to have all the qualities we require in an alternative tonal system. We present the Cayley graph with generators $\{9,5,4\}$ in figure 15. In the same manner as before, we attempt to trace out the diatonic major scale in figure 15 as well. However, we are unsuccessful, primarily because starting at $\frac{3(16)}{4} - 1 = 11$, we only generate $\frac{2(16)}{4} - 1 = 7$ distinct notes before coming back to 11.

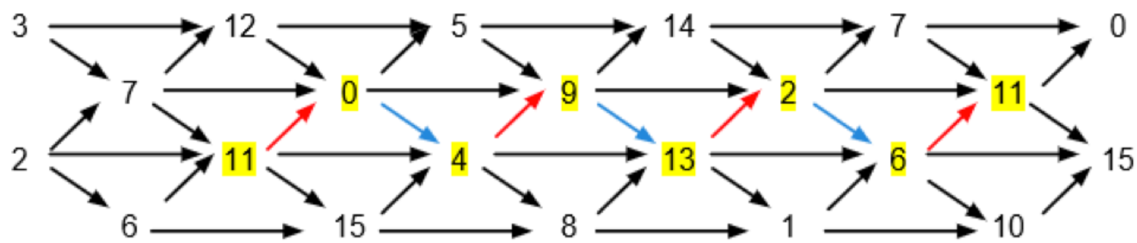

Figure 15: Cayley graph for $\mathbb{Z}_{16}$ with $S = \{9, 5, 3\}$

Therefore the 16-note tonal system does not fulfill the requirement that all notes within the diatonic major scale can be generated with altering steps of $\frac{16}{4} = 4$ and $\frac{16}{4} + 1 = 5$, so it is not a good tonal system that is similar to the 12-note tonal system. Also note that the M3, which is 5 semitones apart from the starting note states that a major 3rd above C is E♯. This does not make sense musically because E♯ is not part of Cs diatonic major scale. Similarly, a m3 above C can be found by counting up 4 semitones which leads to the note E. This also does not make sense because E is in C's diatonic major scale, even though as a m3 above C, it should not be. Therefore it makes sense musically why the 16-note tonal system would not be a good alternative tonal system.

9 Conclusions

Ultimately, we have found that in order for a tonal system be an appropriate alternative to the 12-note tonal system, it must (1) have $\frac{n}{2} + 1$ notes in the diatonic major scale, (2) contain chords that serve the equivalent purposes, (3) contain parallels to the perfect fourth, perfect fifth, major third, and minor third, and (4) be composed of n tones where n is divisible by 4, but not divisible by 8. Although other tonal systems exist, they cannot meet the criteria we set in order to insure that they are similar to the 12-note tonal system that forms most western music.

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